Excitability and pattern formation in a liquid crystal Fabry–Pérot interferometer

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Abstract

We demonstrate a nonlinear Fabry–Pérot interferometer with a nematic liquid crystal to be an optical excitable system. The excitability is shown to arise from the competition of two Kerr type nonlinearities resulting from field-induced molecular reorientation and thermal absorption in the presence of laser radiation. We have observed all the essential features of excitability in this system when it is spatially extended, namely, propagating pulses in one-dimensional space and spiral waves in two dimensions. We have further studied patterns resulting from random noise excitations in this system. © 2001 Elsevier Science B.V. All rights reserved.

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Excitable patterns have long been studied in chemical and biological systems, such as Belousov–Zhabotinsky reaction [1] and cardiac tissues [2]. Recently excitable behavior has also been demonstrated in physical systems, i.e., in liquid crystals [3–5], a laser with external injection [6] and a nonlinear optical cavity [7,8]. In this paper, we investigate optical excitability of nematic liquid crystals (NLCs) in a Fabry–Pérot interferometer. Optical nonlinearities in NLCs are the consequences of field-induced molecular reorientation or thermal absorption in the presence of laser radiation. Due to their high nonlinearities, NLCs have been widely used as nonlinear media in the studies of nonlinear optical phenomena. Optical bistability, self-oscillation and chaos have been observed with nematic film inside Fabry–Pérot interferometer irradiated with a laser beam [9,10]. The dynamics observed in NLCs is shown to result from the interplay of two counteracting mechanisms for nonlinearities. We show that the excitability in NLCs to arise from the competition of these two types of nonlinearities on different time scales. We have observed all the essential features of excitability in this system when it is spatially extended, namely, propagating pulses in one-dimensional space and spiral waves in two dimensions. We have further studied patterns resulting from random noise excitations in this system.

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The interactions of laser radiation with NLCs in a Fabry–Pérot cavity can be described by the classic formula of Fabry–Pérot resonators with the Kerr nonlinearity. In the presence of the two types of nonlinearities, the dynamics of the transversely extended system is governed by the following equations [9,10]:

\[
\tau_1 \frac{dx}{dt} = -x + A \left[ 1 + F \sin^2 \left( \frac{P + x + y}{2} \right) \right], \quad (1a)
\]

\[
\tau_2 \frac{dy}{dt} = -y + B \left[ 1 + F \sin^2 \left( \frac{P + x + y}{2} \right) \right] + D \nabla_x^2 y, \quad (1b)
\]

where \(x\) and \(y\) are the round-trip phase shifts induced by the two mechanisms of laser field-induced molecular reorientation and laser heating, respectively. \(A = x_1 K_{in} T_0\) and \(B = x_2 K_{in} T_0\) are two normalized constants, where \(K\) is a constant which is related to the dispersion of the Fabry–Pérot interferometer [9], \(T_0\) the peak transmission and \(L_{in}\) the input intensity. \(x_1\) and \(x_2\) are constants which measure the strength of the two mechanisms responsible for the phase shifts, respectively. \(\tau_1\) and \(\tau_2\) are their relaxation times. \(F\) is the finesse of the interferometer and \(P\) the initial round-trip phase shift. \(\nabla_x^2\) is the transverse Laplacian describing the heat transportation of the fast variable in the transverse space and \(D\) the diffusion constant. The transverse coupling of the molecular reorientation of the NLC is weaker, compared with the heat diffusion, its effect on the slow variable is therefore negligible.

The round-trip phase shifts due to the laser-induced molecular reorientation and the laser-induced heating are in opposite directions. When the laser radiation increases, the phase shift increases from the former effect, i.e., \(x_1 > 0\), but decreases due to the latter, \(x_2 < 0\). The former process is much slower (about two orders slower) than the latter, therefore \(\tau_1 \gg \tau_2\). The steady state solutions of Eqs. (1a) and (1b) are determined by the intersection points of the two nullclines,

\[x = A \left[ 1 + F \sin^2 \left( \frac{P + x + y}{2} \right) \right]\]

and

\[y = B \left[ 1 + F \sin^2 \left( \frac{P + x + y}{2} \right) \right].\]

Operated in different parameter regions, the system gives an S-shape solution for the fast variable \(y\) and a mirrored C for the slow \(x\) in the \((x, y)\) space. When the two curves intersect at a single point lying on the lower (or upper) branch of the S-shaped bistability curve near the right-hand side vertex, as shown in Fig. 1(a), the system is excitable for \(\varepsilon = \tau_2/\tau_1 \ll 1\). The intersection position of the two nullclines can be adjusted by changing the system parameters \(x_1\), \(x_2\) and \(p\), which correspond to the bias magnetic field across the NLC sample, the sample temperature and the thickness of the NLC film, respectively.

The mechanism of the excitable behavior can be schematically explained in the \((x, y)\) phase space as shown in Fig. 1(a). While the steady state (point A) is linearly stable in the sense that it immediately relaxes back to its original state for a small perturbation, it is however unstable once a larger excitation, referred to as the superthreshold perturbation, pushes this state to a position across the middle branch of the bistable curve, denoted as B in Fig. 1(a). For the latter case, the system first quickly switches to the high branch of the bistable curve, then slowly follows this branch to the left vertex, quickly jumps back to the low branch, and eventually slowly relaxes back to A along the lower branch (refractory period), forming a long excursion in the phase space. The value of the superthreshold perturbation on the fast variable is therefore defined to be the distance between the steady state (at the low branch) and the middle branch point in the vertical direction. The system can also be switched to the high branch of the bistable curve by a perturbation exerted to the slow variable if the excitation pushes the steady state to a position across the right-hand side vertex between the low branch and the middle branch of the bistable curve. In this case, the value of the superthreshold perturbation on the slow variable is therefore defined to be the distance between the steady state and the right-hand side vertex in the horizontal direction. A typical time pulse signal of both the slow and fast variables under a superthreshold perturbation is shown in Fig. 1(b) and (c). The long excursion in space or pulse in time is independent of the external perturbation signal once it is a superthreshold perturbation. During
Fig. 1. (a) Slow x (mirrored C dashed curve) and fast y (S-shaped solid curve) nullclines. The parameter values are $A = 6.3$, $B = -5$, $F = 2$, $p = 5.1$. (b) and (c) are excitation pulses of the $x$ and $y$ variables, respectively, where the excitation signal is applied to the slow variable. The time is normalized to $\tau_2$ and $\varepsilon = \tau_2/\tau_1 = 0.01$. These parameter values are fixed in the following figures.

the period of the intrinsically determined long excursion, including the refractory tail, the system is not susceptible to small external perturbations. Only when the system sufficiently approaches the steady state A, it becomes ready for the next excitation circle. Because of this characteristic feature of excitability, a nearly periodic pulse series can be obtained, as shown in Fig. 2, when we use a white noise as the perturbation signal. This phenomenon, which has been initially investigated in the FitzHugh–Nagumo model [11,12], may have potential applications for generating a coherent resonance optical output with a stochastic input signal.

We now study the spatiotemporal solutions of Eqs. (1a) and (1b). First we consider only one space dimension. Numerical simulations were performed using a split step spectral method. Pulse solutions are found to form when a superthreshold perturbation signal is exerted to the homogeneous state. Typical pulses are shown in Fig. 3. The three states that the system undergoes in one circle of excitability, namely excited (along the upper branch of the bistable curve), refractory (along the
lower branch of the bistable curve), and rest (at the steady state), are readily identified in the amplitude of $y$. When two pulses traveling in the opposite directions collide they annihilate each other, so they are solitary waves. In the two-dimensional case, the integrations were performed on a $512 \times 512$ grid with a box size of 1600 normalized to the square root of the diffusion constant $D$. In two-dimensional excitable systems, the most common patterns are rotating spiral waves which have been extensively investigated [13] both theoretically and experimentally in chemical and biological systems. Such waves have also been obtained in our system. They form whenever a free edge of a propagating wave front is created. The two edges of a truncated planar traveling wave evolve into a pair of spiral waves as shown in Fig. 4. The outer waves eventually become expanding targets resulting from the collision (and then the annihilation) of the outer parts of the two spirals.

An interesting situation arises when the initial condition of the system are given as random Gaussian white noise, either superimposed on the homogeneous steady state or from zero intensity background. We find that different patterns may evolve from different intensity levels of initial noise excitations. When the noise is very weak (below
excitation threshold) the system cannot be excited and simply relaxes to its homogeneous steady state after a short period. As we increase the noise level, first we find transient target-like patterns as shown in Fig. 5. In this case, imperfect targets (Fig. 5(b)) are found to emerge from the noise seeding at the early stage. As they expand they evolve to almost circular wave fronts (Fig. 5(c)) until collisions occur. For the latter cases the segments of the wave fronts that collide annihilate whereas the rest merge (Fig. 5(c) and (d)). The expanding wave fronts eventually collide each other and disappear, so the system returns to the homogeneous steady state (Fig. 5(e)–(h)). However, if the noise level exceeds a certain value the situation changes drastically as shown in Fig. 6. A key difference between the two cases is that for the latter a 'bud' is created within each target when it expands. Each bud is a wave-emitting source generating wave fronts with a certain spatial wavelength in free space. While the outer wave fronts collide and

![Fig. 3.](image)

Fig. 3. (a) Slow and (b) fast propagating pulses in one spatial dimension. The space is normalized to the diffusion constant $D$.

![Fig. 4.](image)

Fig. 4. Temporal evolution of a truncated wave front to a spiral pair (slow variable): (a) $t = 0$, (b) $t = 500$, (c) $t = 1000$, and (d) $t = 3500$. 
annihilate new fronts are generated successively by the buds. As a result it forms a temporally dy-

Fig. 5. Transient target patterns (slow variable) excited by an initial white noise superimposed on the homogeneous state of both the slow and the fast variables. The standard deviation of the noise is 5. The time is (a) t = 0, (b) t = 400, (c) t = 500, (d) t = 600, (e) t = 700, (f) t = 950, (g) t = 1200, and (h) t = 1950.

Fig. 6. Temporal evolution of the spatially quasi-stable spiral patterns (slow variable) from an initial white noise superimposed on the homogeneous state. The standard deviation of the noise is 15.6. The time is (a) t = 0, (b) t = 600, (c) t = 700, (d) t = 1200, (e) t = 2200, and (f) t = 4200.

namical and spatially quasi-stable spiral structure. If we further increase the noise level the number of spirals increases and their spatial distribution is random as shown in Fig. 7(a). Fig. 7(b) shows the power spectrum obtained from the spatial fast Fourier transform of this pattern. Its wide-band feature gives a further indication of the spatial randomness. However, the pattern evolves periodically in time, the periodicity of the temporal waveform at the center of the pattern is clearly seen in Fig. 7(c) and (d). This feature has also been verified by observing the pattern movie
frame by frame. In addition, we have also obtained similar spatial turbulent state by superimposing white noise to the regular spiral waves shown in Fig. 4.

In summary, we have shown that a NLC in a Fabry–Pérot cavity is an optically excitable system in a certain parameter range. We have observed all the essential features of excitablity in this system when it is spatially extended, namely, propagating pulses in one-dimensional space and spiral waves in two dimensions. From random noise excitations transient target patterns, temporally dynamical and spatially quasi-stable spiral patterns, and spatially turbulent patterns have been observed.

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